# The Beckman-Quarles theorem for continuous mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$

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**Summary.** Let $\varphi_n : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $\varphi_n((x_1,...,x_n),(y_1,...,y_n)) = \sum\limits_{i=1}^{n}(x_i - y_i)^2$. We say that $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves distance $d \in \mathbb{C}$, if for each $X, Y \in \mathbb{C}^n$ $\varphi_n(X,Y) = d^2$ implies $\varphi_n(f(X),f(Y)) = d^2$. We prove: if $n \geq 2$ and a continuous $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ has a form $I \circ (\underbrace{\rho, ..., \rho}_{n-\text{times}})$, where $I : \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part and $\rho : \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugation. For $n \geq 3$ and bijective $f$ the theorem follows from Theorem 2 in [8].

The classical Beckman-Quarles theorem states that each unit-distance preserving mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($n \geq 2$) is an isometry, see [1]-[5]. Let $\varphi_n : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $\varphi_n((x_1,...,x_n),(y_1,...,y_n)) = \sum\limits_{i=1}^{n}(x_i - y_i)^2$. We say that $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves distance $d \in \mathbb{C}$, if for each $X, Y \in \mathbb{C}^n$ $\varphi_n(X,Y) = d^2$ implies $\varphi_n(f(X),f(Y)) = d^2$. If $f : \mathbb{C}^n \to \mathbb{C}^n$ and for each $X, Y \in \mathbb{C}^n$ $\varphi_n(X,Y) = \varphi_n(f(X),f(Y))$, then $f$ is an affine mapping with orthogonal linear part; it follows from a general theorem proved in [3, 58 ff], see also [4, p. 30].

Let $D(\mathbb{R}^n, \mathbb{C}^n)$ denote the set of all positive numbers $d$ with the property:

if $X, Y \in \mathbb{R}^n$ and $\varphi_n(X,Y) = d^2$, then there exists a finite set $S_{XY}$ with $\{X,Y\} \subseteq S_{XY} \subseteq \mathbb{R}^n$ such that any map $f : S_{XY} \to \mathbb{C}^n$ that preserves unit distance satisfies $\varphi_n(X,Y) = \varphi_n(f(X),f(Y))$.

Obviously, if $d \in D(\mathbb{R}^n, \mathbb{C}^n)$ and $f : \mathbb{R}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ preserves distance $d$.

Let $D(\mathbb{C}^n, \mathbb{C}^n)$ denote the set of all positive numbers $d$ with the property:

if $X, Y \in \mathbb{C}^n$ and $\varphi_n(X,Y) = d^2$, then there exists a finite set $S_{XY}$ with $\{X,Y\} \subseteq S_{XY} \subseteq \mathbb{C}^n$ such that any map $f : S_{XY} \to \mathbb{C}^n$ that preserves unit distance satisfies $\varphi_n(X,Y) = \varphi_n(f(X),f(Y))$.

Obviously, if $d \in D(\mathbb{C}^n, \mathbb{C}^n)$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ preserves distance $d$.

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If $d > 0$, $X, Y \in \mathbb{C}^n$ and $\varphi_n(X,Y) = d^2$, then there exists an affine $I : \mathbb{C}^n \to \mathbb{C}^n$ such that $I((0, \underbrace{0, \ ..., \ 0}_{n-1 \text{ times}})) = X$, $I((d, \underbrace{0, \ ..., \ 0}_{n-1 \text{ times}})) = Y$, and a linear part of $I$ is orthogonal. Hence,

**(1)** $D(\mathbb{R}^n, \mathbb{C}^n) \subseteq D(\mathbb{C}^n, \mathbb{C}^n)$.

The author proved in [9]:

**(2)** $D(\mathbb{R}^n, \mathbb{C}^n)$ is a dense subset of $(0, \infty)$ for each $n \geq 2$.

From **(2)** we obtain ([9]):

**(3)** if $n \geq 2$ and a continuous $f : \mathbb{R}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ preserves all positive distances.

Obviously,

**(4)** if $n \geq 1$ and $f : \mathbb{R}^n \to \mathbb{C}^n$ preserves all positive distances, then there exists an affine $I : \mathbb{C}^n \to \mathbb{C}^n$ such that a linear part of $I$ is orthogonal and $I_{|\mathbb{R}^n} = f$.

By **(1)** and **(2)**:

**(5)** $D(\mathbb{C}^n, \mathbb{C}^n)$ is a dense subset of $(0, \infty)$ for each $n \geq 2$.

As a corollary of **(5)** we obtain (cf. **(3)**):

**Lemma 1.** If $n \geq 2$ and a continuous $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ preserves all positive distances.

*Proof.* Let $d > 0$, $X, Y \in \mathbb{C}^n$, $\varphi_n(X,Y) = d^2$. By **(5)** there exists a sequence $\{d_m\}_{m=1,2,3,...}$ tending to $d$, where all $d_m$ belong to $D(\mathbb{C}^n, \mathbb{C}^n)$. Since $f$ and $\varphi_n$ are continuous, and $f$ preserves all distances $d_m$ $(m = 1, 2, 3, ...)$,

$$\varphi_n(f(X), f(Y)) = \lim_{m \to \infty} \varphi_n\left(f\left(\frac{d_m}{d}X\right), f\left(\frac{d_m}{d}Y\right)\right) =$$
$$\lim_{m \to \infty} \varphi_n\left(\frac{d_m}{d}X, \frac{d_m}{d}Y\right) = \lim_{m \to \infty} d_m^2 = d^2.$$

Let $\tau : \mathbb{C} \to \mathbb{C}$ denote the complex conjugation.

**Theorem 1.** If $n \geq 2$, $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance and $f_{|\mathbb{R}^n} = \mathrm{id}(\mathbb{R}^n)$, then $f(X) \in \{X, (\underbrace{\tau, ..., \tau}_{n-\text{times}})(X)\}$ for each $X \in \mathbb{C}^n$.

*Proof.* It is true if $X \in \mathbb{R}^n$. Assume now that $X = (x_1, ..., x_n) \in \mathbb{C}^n \setminus \mathbb{R}^n$. Let $x_1 = a_1 + b_1 \cdot \boldsymbol{i}$, ..., $x_n = a_n + b_n \cdot \boldsymbol{i}$, where $a_1, b_1$, ..., $a_n, b_n \in \mathbb{R}$. We choose $j \in \{1, ..., n\}$ with $b_j \neq 0$. For $k \in \{1, ..., n\} \setminus \{j\}$ and $t \in \mathbb{R}$ we define $S_k(t) = (s_{k,1}(t), ..., s_{k,n}(t)) \in \mathbb{R}^n$ as follows:
$s_{k,j}(t) = a_j + tb_k$,
$s_{k,k}(t) = a_k - tb_j$,
$s_{k,i}(t) = a_i$, if $i \in \{1, ..., n\} \setminus \{j, k\}$.

Let $t_0 = \sqrt{\dfrac{\sum\limits_{i=1}^{n} b_i^2}{b_j^2}}$. For each $k \in \{1, ..., n\} \setminus \{j\}$ and all $t \in \mathbb{R}$

$$\varphi_n((x_1, ..., x_n), S_k(t)) = t^2(b_j^2 + b_k^2) - \sum_{i=1}^{n} b_i^2.$$

By this, for each $k \in \{1, ..., n\} \setminus \{j\}$ and all $t \geq t_0$

$$\varphi_n((x_1, ..., x_n), S_k(t)) \geq 0.$$

By **(5)**:

**(6)** for each $k \in \{1, ..., n\} \setminus \{j\}$ the set

$$A_k(x_1, ..., x_n) := \{t \in \mathbb{R} : \varphi_n((x_1, ..., x_n), S_k(t)) \in D(\mathbb{C}^n, \mathbb{C}^n)\}$$

is a dense subset of $(t_0, \infty)$.

Let $f((x_1, ..., x_n)) = (y_1, ..., y_n)$. Since $f$ preserves all distances in $D(\mathbb{C}^n, \mathbb{C}^n)$, for each $k \in \{1, ..., n\} \setminus \{j\}$ and all $t \in A_k(x_1, ..., x_n)$

$$\varphi_n((x_1, ..., x_n), S_k(t)) = \varphi_n(f((x_1, ..., x_n)), f(S_k(t))) = \varphi_n((y_1, ..., y_n), S_k(t)).$$

Hence, for each $k \in \{1, ..., n\} \setminus \{j\}$ and all $t \in A_k(x_1, ..., x_n)$

$$t^2(b_j^2 + b_k^2) - \sum_{i=1}^{n} b_i^2 = (y_j - a_j - tb_k)^2 + (y_k - a_k + tb_j)^2 + \sum_{i \in \{1,...,n\} \setminus \{j,k\}} (y_i - a_i)^2.$$

Thus, for each $k \in \{1, ..., n\} \setminus \{j\}$ and all $t \in A_k(x_1, ..., x_n)$

$$\sum_{i=1}^{n} (y_i - a_i)^2 + \sum_{i=1}^{n} b_i^2 = 2t \cdot (b_k(y_j - a_j) - b_j(y_k - a_k)).$$

Hence by **(6)**:

**(7)** $y_k - a_k = \dfrac{b_k}{b_j} \cdot (y_j - a_j)$ for each $k \in \{1, ..., n\} \setminus \{j\}$

and

**(8)** $\sum\limits_{i=1}^{n} (y_i - a_i)^2 + \sum\limits_{i=1}^{n} b_i^2 = 0.$

Applying **(7)** to **(8)** we obtain

$$(y_j - a_j)^2 + \sum_{k \in \{1,...,n\} \setminus \{j\}} \frac{b_k^2}{b_j^2} \cdot (y_j - a_j)^2 + \sum_{i=1}^{n} b_i^2 = 0.$$

It gives $\left(\dfrac{(y_j - a_j)^2}{b_j^2} + 1\right) \cdot \sum\limits_{i=1}^{n} b_i^2 = 0$. Since $\sum\limits_{i=1}^{n} b_i^2 \neq 0$, we get

$$\underbrace{y_j = a_j + b_j \cdot \boldsymbol{i} = x_j}_{\text{case 1}} \quad \text{or} \quad \underbrace{y_j = a_j - b_j \cdot \boldsymbol{i} = \tau(x_j)}_{\text{case 2}}.$$

In case 1, by **(7)** for each $k \in \{1, ..., n\} \setminus \{j\}$

$$y_k = a_k + \frac{b_k}{b_j} \cdot (y_j - a_j) = a_k + \frac{b_k}{b_j} \cdot (a_j + b_j \cdot \boldsymbol{i} - a_j) = a_k + b_k \cdot \boldsymbol{i} = x_k.$$

In case 2, by **(7)** for each $k \in \{1, ..., n\} \setminus \{j\}$

$$y_k = a_k + \frac{b_k}{b_j} \cdot (y_j - a_j) = a_k + \frac{b_k}{b_j} \cdot (a_j - b_j \cdot \boldsymbol{i} - a_j) = a_k - b_k \cdot \boldsymbol{i} = \tau(x_k).$$

The proof is completed.

Let $n \geq 2$, $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, $f_{|\mathbb{R}^n} = \mathrm{id}(\mathbb{R}^n)$. We would like to prove: $f = \mathrm{id}(\mathbb{C}^n)$ or $f = (\underbrace{\tau, ..., \tau}_{n-\text{times}})$; we will prove it later in Theorem 2.

By Theorem 1 the sets

$$\boldsymbol{A} = \{X \in \mathbb{C}^n : f(X) = X\}$$

and

$$\boldsymbol{B} = \{X \in \mathbb{C}^n : f(X) = (\underbrace{\tau, ..., \tau}_{n-\text{times}})(X)\}$$

satisfy $\boldsymbol{A} \cup \boldsymbol{B} = \mathbb{C}^n$.

Let $\psi_n : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, $\psi_n((x_1, ..., x_n), (y_1, ..., y_n)) = \sum\limits_{k=1}^{n} \mathrm{Im}(x_k) \cdot \mathrm{Im}(y_k)$.

**Lemma 2.** If $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{C}$, $\varphi_n((x_1, ..., x_n), (y_1, ..., y_n)) = 1$ and $\psi_n((x_1, ..., x_n), (y_1, ..., y_n)) \neq 0$, then

**(9)** $$(y_1, ..., y_n) \in \boldsymbol{A} \text{ implies } (x_1, ..., x_n) \in \boldsymbol{A}$$

and

**(10)** $$(y_1, ..., y_n) \in \boldsymbol{B} \text{ implies } (x_1, ..., x_n) \in \boldsymbol{B}.$$

*Proof.* We prove only **(9)**, the proof of **(10)** follows analogically. Assume, on the contrary, that $(y_1, ..., y_n) \in \boldsymbol{A}$ and $(x_1, ..., x_n) \notin \boldsymbol{A}$. Since $\boldsymbol{A} \cup \boldsymbol{B} = \mathbb{C}^n$, $(x_1, ..., x_n) \in \boldsymbol{B}$. Let $x_1 = a_1 + b_1 \cdot \boldsymbol{i}$, ..., $x_n = a_n + b_n \cdot \boldsymbol{i}$, $y_1 = \widetilde{a}_1 + \widetilde{b}_1 \cdot \boldsymbol{i}$, ..., $y_n = \widetilde{a}_n + \widetilde{b}_n \cdot \boldsymbol{i}$, where $a_1, b_1, ..., a_n, b_n, \widetilde{a}_1, \widetilde{b}_1, ..., \widetilde{a}_n, \widetilde{b}_n \in \mathbb{R}$. Since $f$ preserves unit distance,

**(11)** $$\varphi_n((x_1, ..., x_n), (y_1, ..., y_n)) =$$

$$\varphi_n(f((x_1, ..., x_n)), f((y_1, ..., y_n))) = \sum_{k=1}^{n} (a_k - b_k \cdot \boldsymbol{i} - \widetilde{a}_k - \widetilde{b}_k \cdot \boldsymbol{i})^2.$$

Obviously,

**(12)** $$\varphi_n((x_1, ..., x_n), (y_1, ..., y_n)) = \sum_{k=1}^{n} (a_k + b_k \cdot \boldsymbol{i} - \widetilde{a}_k - \widetilde{b}_k \cdot \boldsymbol{i})^2.$$

Subtracting **(11)** and **(12)** by sides we obtain

$$4 \sum_{k=1}^{n} b_k \widetilde{b}_k + 4 \sum_{k=1}^{n} b_k (a_k - \widetilde{a}_k) \cdot \boldsymbol{i} = 0,$$

so in particular $\psi_n((x_1,...,x_n),(y_1,...,y_n)) = \sum\limits_{k=1}^{n} b_k \widetilde{b}_k = 0$, a contradiction.

The next lemma is obvious.

**Lemma 3.** For each $S,T \in \mathbb{R}^n$ there exist $m \in \{1,2,3,...\}$ and $P_1,...,P_m \in \mathbb{R}^n$ such that $||S-P_1|| = ||P_1-P_2|| = ... = ||P_{m-1}-P_m|| = ||P_m-T|| = 1$.

**Lemma 4.** For each $X \in \mathbb{C}^n \setminus \mathbb{R}^n$

$$(\underbrace{\boldsymbol{i},...,\boldsymbol{i}}_{n-\text{times}}) \in \boldsymbol{A} \text{ implies } X \in \boldsymbol{A}$$

and

$$(\underbrace{\boldsymbol{i},...,\boldsymbol{i}}_{n-\text{times}}) \in \boldsymbol{B} \text{ implies } X \in \boldsymbol{B}.$$

*Proof.* Let $X = (a_1 + b_1 \cdot \boldsymbol{i},\ ...,\ a_n + b_n \cdot \boldsymbol{i})$, where $a_1,...,a_n,b_1,...,b_n \in \mathbb{R}$. We choose $j \in \{1,...,n\}$ with $b_j \neq 0$. The points

$$S = \Bigg(a_1 + \sqrt{\frac{1+(b_j-1)^2}{n-1}},\ ...,\ a_{j-1} + \sqrt{\frac{1+(b_j-1)^2}{n-1}}\ ,$$
$$\underbrace{a_j + \sqrt{1 + \sum_{k \in \{1,...,n\} \setminus \{j\}} b_k^2}}_{j-\text{th coordinate}}\ ,$$
$$a_{j+1} + \sqrt{\frac{1+(b_j-1)^2}{n-1}},\ ...,\ a_n + \sqrt{\frac{1+(b_j-1)^2}{n-1}}\Bigg)$$

and $T = (0,\ ...,\ 0, \underbrace{\sqrt{n}}_{j-\text{th coordinate}}, 0,\ ...,\ 0)$ belong to $\mathbb{R}^n$. Applying Lemma 3 we find $m \in \{1,2,3,...\}$ and $P_1,...,P_m \in \mathbb{R}^n$ satisfying $||S-P_1|| = ||P_1-P_2|| = ... = ||P_{m-1}-P_m|| = ||P_m-T|| = 1$. The points

$X_1 = X$,

$$X_2 = \left(a_1,\ ...,\ a_{j-1},\ \underbrace{a_j + \sqrt{1 + \sum_{k \in \{1,...,n\} \setminus \{j\}} b_k^2} + b_j \cdot \boldsymbol{i},}_{j-\text{th coordinate}}\ a_{j+1},\ ...,\ a_n\right),$$

$$X_3 = S + \left(0,\ ...,\ 0, \underbrace{\boldsymbol{i}}_{j-\text{th coordinate}}, 0,\ ...,\ 0\right) =$$

$$\left(a_1+\sqrt{\frac{1+(b_j-1)^2}{n-1}},\ \ldots,\ a_{j-1}+\sqrt{\frac{1+(b_j-1)^2}{n-1}}\ ,\right.$$

$$\underbrace{a_j+\sqrt{1+\sum_{k\in\{1,\ldots,n\}\setminus\{j\}} b_k^2}\ +\ \boldsymbol{i}}_{j-\text{th coordinate}}\ ,$$

$$\left. a_{j+1}+\sqrt{\frac{1+(b_j-1)^2}{n-1}},\ \ldots,\ a_n+\sqrt{\frac{1+(b_j-1)^2}{n-1}}\right),$$

$$X_4=P_1+(0,\ \ldots,\ 0,\ \underbrace{\boldsymbol{i}}_{j-\text{th coordinate}}\ ,0,\ \ldots,\ 0),$$

$$X_5=P_2+(0,\ \ldots,\ 0,\ \underbrace{\boldsymbol{i}}_{j-\text{th coordinate}}\ ,0,\ \ldots,\ 0),$$

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$$X_{m+3}=P_m+(0,\ \ldots,\ 0,\ \underbrace{\boldsymbol{i}}_{j-\text{th coordinate}}\ ,0,\ \ldots,\ 0),$$

$$X_{m+4}=T+(0,\ \ldots,\ 0,\ \underbrace{\boldsymbol{i}}_{j-\text{th coordinate}}\ ,0,\ \ldots,\ 0)=(0,\ \ldots,\ 0,\ \underbrace{\sqrt{n}+\boldsymbol{i}}_{j-\text{th coordinate}},0,\ \ldots,\ 0),$$

$$X_{m+5}=(\underbrace{\boldsymbol{i},\ldots,\boldsymbol{i}}_{n-\text{times}})$$

belong to $\mathbb{C}^n$ and satisfy:

$\varphi_n(X_{k-1},X_k)=1$ for each $k\in\{2,3,\ldots,m+5\}$, $\psi_n(X_1,X_2)=b_j^2\neq 0$, $\psi_n(X_2,X_3)=b_j\neq 0$, $\psi_n(X_{k-1},X_k)=1$ for each $k\in\{4,5,\ldots,m+5\}$.

By Lemma 2 for each $k\in\{2,3,\ldots,m+5\}$

$$X_k\in\boldsymbol{A}\ \text{implies}\ X_{k-1}\in\boldsymbol{A}$$

and

$$X_k\in\boldsymbol{B}\ \text{implies}\ X_{k-1}\in\boldsymbol{B}.$$

Therefore, $(\underbrace{\boldsymbol{i},\ldots,\boldsymbol{i}}_{n-\text{times}})=X_{m+5}\in\boldsymbol{A}$ implies $X=X_1\in\boldsymbol{A}$, and also, $(\underbrace{\boldsymbol{i},\ldots,\boldsymbol{i}}_{n-\text{times}})=X_{m+5}\in\boldsymbol{B}$ implies $X=X_1\in\boldsymbol{B}$.

**Theorem 2.** If $n\geq 2$, $f:\mathbb{C}^n\to\mathbb{C}^n$ preserves unit distance and $f_{|\mathbb{R}^n}=\mathrm{id}(\mathbb{R}^n)$, then $f=\mathrm{id}(\mathbb{C}^n)$ or $f=(\underbrace{\tau,\ldots,\tau}_{n-\text{times}})$.

*Proof.* By Lemma 4

$$(\underbrace{\boldsymbol{i},\ldots,\boldsymbol{i}}_{n-\text{times}})\in\boldsymbol{A}\ \ \text{implies}\ \ \mathbb{C}^n\setminus\mathbb{R}^n\subseteq\boldsymbol{A}$$

and

$$(\underbrace{\boldsymbol{i}, ..., \boldsymbol{i}}_{n-\text{times}}) \in \boldsymbol{B} \ \text{ implies } \ \mathbb{C}^n \setminus \mathbb{R}^n \subseteq \boldsymbol{B}.$$

Obviously, $\mathbb{R}^n \subseteq \boldsymbol{A}$ and $\mathbb{R}^n \subseteq \boldsymbol{B}$. Therefore,

$$\boldsymbol{A} = \mathbb{C}^n \text{ and } f = \mathrm{id}(\mathbb{C}^n), \ \text{ if } (\underbrace{\boldsymbol{i}, ..., \boldsymbol{i}}_{n-\text{times}}) \in \boldsymbol{A},$$

and also,

$$\boldsymbol{B} = \mathbb{C}^n \text{ and } f = (\underbrace{\tau, ..., \tau}_{n-\text{times}}), \ \text{ if } (\underbrace{\boldsymbol{i}, ..., \boldsymbol{i}}_{n-\text{times}}) \in \boldsymbol{B}.$$

Theorem 2 has a simpler proof under the additional assumption that $f$ is continuous. We need a topological lemma.

**Lemma 5.** $\mathbb{C}^n \setminus \mathbb{R}^n$ is connected for each $n \geq 2$.

*Proof.* Let $X = (a_1 + b_1 \cdot \boldsymbol{i}, \ ..., \ a_n + b_n \cdot \boldsymbol{i}) \in \mathbb{C}^n \setminus \mathbb{R}^n$, where $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$. We choose $j \in \{1, ..., n\}$ with $b_j \neq 0$. Then

$$Y := \left( 0, \ ..., \ 0, \ \underbrace{\frac{b_j}{|b_j|} \cdot \boldsymbol{i}}_{j-\text{th coordinate}}, 0, \ ..., \ 0 \right) \in \mathbb{C}^n \setminus \mathbb{R}^n$$

and the segments $(\underbrace{\boldsymbol{i}, ..., \boldsymbol{i}}_{n-\text{times}})Y$ and $YX$ are disjoint from $\mathbb{R}^n$. These segments form a path joining $(\underbrace{\boldsymbol{i}, ..., \boldsymbol{i}}_{n-\text{times}})$ and $X$, $X$ is an abitrary point in $\mathbb{C}^n \setminus \mathbb{R}^n$. It proves that $\mathbb{C}^n \setminus \mathbb{R}^n$ is connected.

Since $\mathrm{id}(\mathbb{C}^n \setminus \mathbb{R}^n)$ and $(\underbrace{\tau, ..., \tau}_{n-\text{times}})_{|\mathbb{C}^n \setminus \mathbb{R}^n}$ are continuous, for continuous $f$ in Theorem 2

$$\boldsymbol{A} \setminus \mathbb{R}^n = \{X \in \mathbb{C}^n \setminus \mathbb{R}^n : f(X) = X\}$$

and

$$\boldsymbol{B} \setminus \mathbb{R}^n = \{X \in \mathbb{C}^n \setminus \mathbb{R}^n : f(X) = (\underbrace{\tau, ..., \tau}_{n-\text{times}})(X)\}$$

are the closed subsets of $\mathbb{C}^n \setminus \mathbb{R}^n$. Obviously,

$$(\boldsymbol{A} \setminus \mathbb{R}^n) \cup (\boldsymbol{B} \setminus \mathbb{R}^n) = \mathbb{C}^n \setminus \mathbb{R}^n$$

and

$$(\boldsymbol{A} \setminus \mathbb{R}^n) \cap (\boldsymbol{B} \setminus \mathbb{R}^n) = \emptyset.$$

Hence by Lemma 5

$$\boldsymbol{A} \setminus \mathbb{R}^n = \mathbb{C}^n \setminus \mathbb{R}^n \quad \text{or} \quad \boldsymbol{B} \setminus \mathbb{R}^n = \mathbb{C}^n \setminus \mathbb{R}^n.$$

Thus,

$$\boldsymbol{A} = \mathbb{C}^n \quad \text{and} \quad f = \text{id}(\mathbb{C}^n)$$

or

$$\boldsymbol{B} = \mathbb{C}^n \quad \text{and} \quad f = (\underbrace{\tau, ..., \tau}_{n-\text{times}}).$$

**Theorem 3.** If $n \geq 2$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves all positive distances, then $f$ has a form $I \circ (\underbrace{\rho, ..., \rho}_{n-\text{times}})$, where $\rho \in \{\text{id}(\mathbb{C}), \tau\}$ and $I : \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part.

*Proof.* By **(4)** there exists an affine $I : \mathbb{C}^n \to \mathbb{C}^n$ such that a linear part of $I$ is orthogonal and $I_{|\mathbb{R}^n} = f_{|\mathbb{R}^n}$. Then $I^{-1} \circ f : \mathbb{C}^n \to \mathbb{C}^n$ preserves all positive distances, $(I^{-1} \circ f)_{|\mathbb{R}^n} = \text{id}(\mathbb{R}^n)$. By Theorem 2 $I^{-1} \circ f = \text{id}(\mathbb{C}^n)$ or $I^{-1} \circ f = (\underbrace{\tau, ..., \tau}_{n-\text{times}})$. In the first case $f = I \circ (\underbrace{\text{id}(\mathbb{C}), ..., \text{id}(\mathbb{C})}_{n-\text{times}})$, in the second case $f = I \circ (\underbrace{\tau, ..., \tau}_{n-\text{times}})$.

As a corollary of Lemma 1 and Theorem 3 we get:

**Theorem 4.** If $n \geq 2$ and a continuous $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ has a form $I \circ (\underbrace{\rho, ..., \rho}_{n-\text{times}})$, where $\rho \in \{\text{id}(\mathbb{C}), \tau\}$ and $I : \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part.

Any bijective $f : \mathbb{C}^n \to \mathbb{C}^n$ $(n \geq 3)$ that preserves unit distance has a form $I \circ (\rho, ..., \rho)$, where $\rho : \mathbb{C} \to \mathbb{C}$ is a field isomorphism and $I : \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part; it follows from Theorem 2 in [8].

The author proved in [12]:

**(13)** each unit-distance preserving mapping from $\mathbb{C}^2$ to $\mathbb{C}^2$ has a form $I \circ (\rho, \rho)$, where $\rho : \mathbb{C} \to \mathbb{C}$ is a field homomorphism and $I : \mathbb{C}^2 \to \mathbb{C}^2$ is an affine mapping with orthogonal linear part.

The first proof of **(13)** in [12] is based on the results of [10] and [11]. The second proof of **(13)** in [12] is based on the result of [7]. Obviously, for $n = 2$ Theorem 3 follows from **(13)**.

If a continuous $\sigma : \mathbb{C} \to \mathbb{C}$ is a field homomorphism, then $(\sigma, \sigma) : \mathbb{C}^2 \to \mathbb{C}^2$ is continuous and preserves unit distance, also $(\sigma, \sigma)((0, 0)) = (0, 0)$, $(\sigma, \sigma)((1, 0)) = (1, 0)$, $(\sigma, \sigma)((0, 1)) = (0, 1)$. Therefore, by Theorem 4 $\sigma = \text{id}(\mathbb{C})$ or $\sigma = \tau$. We have obtained an alternative geometric proof of a well-known result:

**(14)** the only continuous field endomorphisms of $\mathbb{C}$ are $\text{id}(\mathbb{C})$ and $\tau$.

An algebraic proof of **(14)** may be found in [6, Lemma 1, p. 356]. Conversely, for $n = 2$ Theorem 4 follows from **(13)** and **(14)**.

Let $d \in \mathbb{C}\setminus\{0\}$ and $\mathbb{C} \ni x \xrightarrow{\tau_d} \frac{d}{\tau(d)} \cdot \tau(x) \in \mathbb{C}$. As a consequence of Theorem 4 we get:

**Theorem 5.** If $n \geq 2$ and a continuous $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves distance $d \in \mathbb{C} \setminus \{0\}$, then $f$ has a form $I \circ (\underbrace{\rho, ..., \rho}_{n-\text{times}})$, where $\rho \in \{\mathrm{id}(\mathbb{C}), \tau_d\}$ and $I : \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part.

**Corollary.** If $n \geq 2$, $d_1, d_2 \in \mathbb{C} \setminus \{0\}$, $\frac{d_1^2}{d_2^2} \notin \mathbb{R}$ and a continuous $f : \mathbb{C}^n \to \mathbb{C}^n$ preserves distances $d_1$ and $d_2$, then $f$ is an affine mapping with orthogonal linear part.

Theorems 1–4 do not hold for $n = 1$ because the mapping $\mathbb{C} \ni z \longrightarrow z + \mathrm{Im}(z) \in \mathbb{C}$ preserves all real distances. In case $n = 1$, there is an easy result:

**Theorem 6.** Let $f : \mathbb{C} \to \mathbb{C}$ and for each $x, y \in \mathbb{C}$ $\varphi_1(x, y) \in \mathbb{R}$ implies $\varphi_1(x, y) = \varphi_1(f(x), f(y))$. Then $f$ has a form $I \circ \rho$, where $\rho \in \{\mathrm{id}(\mathbb{C}), \tau\}$ and $I : \mathbb{C} \to \mathbb{C}$ is an affine mapping with orthogonal linear part.

# References


[1] F. S. Beckman and D. A. Quarles Jr., *On isometries of euclidean spaces*, Proc. Amer. Math. Soc. 4 (1953), 810–815.

[2] W. Benz, *An elementary proof of the theorem of Beckman and Quarles*, Elem. Math. 42 (1987), 4–9.

[3] W. Benz, *Geometrische Transformationen (unter besonderer Berücksichtigung der Lorentztransformationen)*, BI Wissenschaftsverlag, Mannheim, 1992.

[4] W. Benz, *Real geometries*, BI Wissenschaftsverlag, Mannheim, 1994.

[5] U. Everling, *Solution of the isometry problem stated by K. Ciesielski*, Math. Intelligencer 10 (1988), no. 4, p. 47.

[6] M. Kuczma, *An introduction to the theory of functional equations and inequalities: Cauchy's equation and Jensen's inequality*, Polish Scientific Publishers and Silesian University, Warszawa–Kraków–Katowice, 1985.

[7] H. Schaeffer, *Der Satz von Benz-Radó*, Aequationes Math. 31 (1986), 300–309.

[8] E. M. Schröder, *Zur Kennzeichnung distanztreuer Abbildungen in nichteuklidischen Räumen*, J. Geom. 15 (1980), no. 2, 108–118.

[9] A. Tyszka, *Beckman-Quarles type theorems for mappings from $\mathbb{R}^n$ to $\mathbb{C}^n$*, Aequationes Math. 67 (2004), 225–235.

[10] A. Tyszka, *The Beckman-Quarles theorem for mappings from* $\mathbb{R}^2$ *to* $\mathbb{F}^2$, *where* $\mathbb{F}$ *is a subfield of a commutative field extending* $\mathbb{R}$, Abh. Math. Sem. Univ. Hamburg 74 (2004), 77–87.

[11] A. Tyszka, *A discrete form of the Beckman-Quarles theorem for mappings from* $\mathbb{R}^2$ ($\mathbb{C}^2$) *to* $\mathbb{F}^2$, *where* $\mathbb{F}$ *is a subfield of a commutative field extending* $\mathbb{R}$ ($\mathbb{C}$), J. Geom, to appear.

[12] A. Tyszka, *The Beckman-Quarles theorem for mappings from* $\mathbb{C}^2$ *to* $\mathbb{C}^2$, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 21 (2005), no. 1 (electronic), to appear.

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